Sequential Estimation of Dynamic Programming Models with Unobserved Heterogeneity

Hiroyuki Kasahara
Department of Economics
University of Western Ontario
hkasahar@uwo.ca

Katsumi Shimotsu
Department of Economics
Queen's University
shimotsu@econ.queensu.ca

February 4, 2009

Abstract

This paper develops a new computationally attractive procedure for estimating dynamic discrete choice models that is applicable to a wide range of dynamic programming models. The proposed procedure can accommodate unobserved state variables that (i) are neither additively separable nor follow generalized extreme value distribution, (ii) are serially correlated, and (iii) affect the choice set. Our estimation algorithm sequentially updates the parameter estimate and the value function estimate. It builds upon the idea of the iterative estimation algorithm proposed by Aguirregabiria and Mira (2002, 2007) but conducts iteration using the value function mapping rather than the policy iteration mapping. Its implementation is straightforward in terms of computer programming; unlike the Hotz-Miller type estimators, there is no need to reformulate a fixed point mapping in the value function space as that in the space of probability distributions. It is also applicable to estimate models with unobserved heterogeneity. We analyze the convergence property of our sequential algorithm and derive the conditions for its convergence. We also develop an approximated procedure which reduces computational cost substantially without deteriorating the convergence rate.

Keywords: dynamic discrete choice, value function mapping, nested pseudo likelihood, unobserved heterogeneity.

JEL Classification Numbers: C13, C14, C63.

1 Introduction

Numerous empirical studies have demonstrated that the estimation of dynamic discrete models enhances our understanding of individual and firm behavior and provide important policy implications.[1] The literature on estimating dynamic models of discrete choice was pioneered by Gotz and McCall (1980), Wolpin (1984), Miller (1984), Pakes (1986), and Rust (1987, 1988). Standard methods for estimating infinite horizon dynamic discrete choice models requires repeatedly solving the fixed point problem (i.e., Bellman equation) during optimization and can be very costly when the dimensionality of state space is large.

To reduce the computational burden, Hotz and Miller (1993) developed a simpler two-step estimator, called *Conditional Choice Probability (CCP) estimator*, by exploiting the inverse mapping from the value functions to the conditional choice probabilities.[2] Aguirregabiria and Mira (2002, 2007) developed a recursive extension of the two-step method of Hotz and Miller (1993), called the *nested pseudo likelihood (NPL) algorithm.* The applicability of these Hotz and Miller-type estimators is, however, limited when unobserved state variables are not additively separable and (generalized-) extreme value distributed because evaluating the inverse mapping from the value functions to the conditional choice probabilities is computationally difficult. Recently, Arcidiacono and Miller (2008) develop estimators that relax some of the limitations of CCP estimator by adapting the Expectation-Maximization (EM) algorithm into the NPL algorithm in estimating models with unobserved heterogeneity; while Arcidiacono and Miller provide important contributions to the literature, little is known about the convergence property of their algorithm and it is not clear how computationally easy it is to apply their estimation method to a model that does not exhibit finite time dependence.

This paper develops a new estimation procedure applicable to a class of infinite horizon dynamic discrete choice models with unobserved state variables that (i) are neither additively separable nor follow generalized extreme value distribution, (ii) are serially correlated, and (iii) affect the choice set. Our estimation method is based on the value function mapping (i.e., Bellman equation) and, hence, unlike the Hotz-Miller type estimators, there is no need to reformulate Bellman equation as a fixed point mapping in the space of probability distributions using the policy iteration operator. This is the major advantage of our method over the Hotz-Miller type estimators because evaluating the policy iteration operator is often difficult without the assumption of additively-separable generalized extreme value distribution.

Our estimation algorithm is analogous to the NPL algorithm [cf., Aguirregabiria and Mira (2002, 2007) and Kasahara and Shimotsu (2008a, 2008b)] but its iteration is based on the value function mapping rather than the policy iteration mapping. Our procedure iterates on the following two-steps. First, given an initial estimator of the value function, we estimate the model's parameter by repeatedly solving a *finite horizon q-period model* in which the (q+1)-

[1] Contributions include Berkovec and Stern (1991), Keane and Wolpin (1997), Rust and Phelan (1997), Rothwell and Rust (1997), Altug and Miller (1998), Gilleskie (1998), Eckstein and Wolpin (1999), Aguirregabiria (1999), Kasahara and Lapham (2008), and Kasahara (2009).

[2] A number of recent papers in empirical industrial organization build on the idea of Hotz and Miller (1993) to develop two-step estimators for models with multiple agents (e.g., Bajari, Benkard, and Levin, 2007; Pakes, Ostrovsky, and Berry, 2007; Pesendorfer and Schmidt-Dengler, 2008; Bajari and Hong, 2006).

th period's value function is given by the initial value function estimate. Second, we update the value function estimate by solving a q-period model under the newly estimated parameter starting from the previous estimate of the value function as the continuation value in the q-th period. This sequential algorithm is computationally easy if we choose a small value of q; if we choose $q = 1$, for instance, then the computational cost of solving this finite horizon model is equivalent to solving a static model. Iterating this procedure generates a sequence of estimators for a pair of parameter and value function. Upon convergence, the limit of this sequence does not depend on an initial value function estimator and, thus, our method is applicable even when an initial consistent estimator of value function is not available.

We analyze the convergence property of our proposed sequential algorithm. The possibility of non-convergence of the NPL algorithm proposed by Aguirregabiria and Mira (2002, 2007) is a concern as illustrated by Pesendorfer and Schmidt-Dengler (2008) and Collard-Wexler (2006).[3] Since our algorithm is very similar to the original NPL algorithm developed by Aguirregabiria and Mira, understanding the convergence property of our sequential algorithm is important. We show that a key determinant of the convergence of our algorithm is the *contraction* property of the value function mapping. By the Blackwell's sufficient condition, the value function mapping is a contraction where a discount factor determines the contraction rate, and iterating the value function mapping improves the contraction property. As a result, our sequential algorithm achieves convergence when we choose sufficiently large q. To reduce computational cost further, we also develop approximation procedure called the approximate q-NPL algorithm. The computational cost of implementing this approximate algorithm is substantially less than the original sequential algorithm but, nonetheless, its first-order convergence rate is the same as that of the original sequential algorithm.

The rest of the paper is organized as follows. Section 2 illustrates the basic idea of our algorithm by using a simple example. Section 3 introduces a class of models we consider in this paper. Section 4 presents our sequential estimation procedure and derive its convergence property. Section 5 reports some simulation results.

2 Example: Machine Replacement Model with Non-Additive Productivity Shocks

To illustrate the basic idea of our estimator, consider the following version of Rust's machine replacement model. Let x_t denote machine age and let the variable $a_t \in \{0, 1\}$ represent the machine replacement decision. Both x_t and a_t are observable to a researcher. There are two state

[3] Pesendorfer and Schmidt-Dengler (2008) provided simulation evidence that the NPL algorithm may not necessarily converge while Collard-Wexler (2006) used the NPL algorithm to estimate a model of entry and exit for the ready-mix concrete industry and found that $\tilde{P}_j$'s "cycle around several values without converging." Kasahara and Shimotsu (2008b) analyze the conditions under which the NPL algorithm achieves convergence and derive its convergence rate.

variables in the model that are not observable to a researcher: an idiosyncratic productivity shock ϵ_t and an choice-dependent cost shock $\xi_t(a_t)$. The profit function is given by $u_\theta(a_t, x_t, \epsilon_t)+\xi_t(a_t)$, where $u_\theta(a_t, x_t, \epsilon_t) = \exp(\theta_1 x_t(1-a_t)+\epsilon_t)-\theta_2 a_t$. Here, $\exp(\theta_1 x_t(1-a_t)+\epsilon_t)$ represents revenue function, θ_1 is the rate of depreciation in machine productivity, and θ_2 is replacement cost. We assume that $\xi_t = (\xi_t(0), \xi_t(1))'$ follows Type 1 extreme value distribution independently across alternatives while ϵ_t is independently drawn from $N(0, \sigma_\epsilon^2)$. The transition function of machine age x_t is given by $x_{t+1} = a_t + (1-a_t)(x_t+1)$.

A firm maximizes the expected discounted sum of revenues, $E[\sum_{j=0}^{\infty}\beta^j(u_\theta(a_{t+j}, x_{t+j}, \epsilon_{t+j}) + \xi_{t+j}(a_{t+j}))|a_t, x_t]$. The Bellman equation for this dynamic optimization problem is written as $W(x, \epsilon, \xi) = \max_{a\in\{0,1\}} u_\theta(a, x, \epsilon) + \xi(a) + \beta \int\int W(a+(1-a)(x+1), \epsilon', \xi')g_\epsilon(d\epsilon'|x)g_\xi(d\xi'|x)$. Define the integrated value function $V(x) = \int\int W(x, \epsilon, \xi)g_\epsilon(d\epsilon'|x)g_\xi(d\xi'|x)$. Then, using the properties of Type 1 extreme value distribution, the integrated Bellman equation is written as:

$$V(x) = \int \left(\gamma + \ln\left(\sum_{a\in\{0,1\}} \exp\left(u_\theta(a, x, \epsilon') + \beta V\left(a+(1-a)(x+1)\right)\right)\right)\right) \frac{\phi(\epsilon'/\sigma_\epsilon)}{\sigma_\epsilon} d\epsilon' \equiv [\Gamma(\theta, V)](x), \tag{1}$$

where $\phi(\cdot)$ is the standard normal probability distribution and γ is Euler's constant. The Bellman operator $\Gamma(\theta, \cdot)$ is defined by the right hand side of this integrated Bellman equation. Denote the fixed point of the integrated Bellman equation (1) by V_θ $[= \Gamma(\theta, V_\theta)]$. The value of $V_\theta(x)$ represents the value of a firm with machine age x. Given V_θ, the conditional choice probability of replacement (i.e., $a = 1$) is given by

$$P_\theta(a=1|x) = \int \left(\frac{\exp(u_\theta(1, x, \epsilon') + \beta V_\theta(1))}{\sum_{a'\in\{0,1\}} \exp\left(u_\theta(a', x, \epsilon') + \beta V_\theta\left(a' + (1-a')(x+1)\right)\right)}\right) \frac{\phi(\epsilon'/\sigma_\epsilon)}{\sigma_\epsilon} d\epsilon' \equiv \Lambda(\theta, V_\theta), \tag{2}$$

while $P_\theta(a=0|x) = 1 - P_\theta(a=1|x)$. Here, the operator defined by the right hand side of (2), denoted by $\Lambda(\theta, \cdot)$, maps the value function space into the probability space.

To estimate the unknown parameter vector θ given a cross sectional data $\{x_i, a_i\}_{i=1}^n$, where n is the sample size, we may use the Rust's NFXP algorithm by repeatedly solving the fixed point of (1) and evaluating the conditional choice probabilities (2) for every candidate value of θ to maximize the likelihood $\sum_{i=1}^n \ln P_\theta(a_i|x_i)$, where the integral with respect to ϵ can be evaluated by quadrature methods or simulations. The NFXP algorithm is costly because it is computationally intensive to solve the fixed point of (1), which requires repeatedly applying the fixed point iteration, $V^j = \Gamma(\theta, V^{j-1})$, starting from an initial guess V^0 until convergence. Estimating this replacement model using the Hotz-Miller type estimators [cf., Hotz and Miller (1993) and Aguirregabiria and Mira (2002, 2007)] is not straightforward either because evaluating the inverse mapping from the value functions to the conditional choice probabilities is computationally difficult due to the presence of normally distributed shocks, ϵ.

We propose a simple alternative estimation method applicable to models with unobserved state variables that are neither additively separable and nor extreme-value distributed. Our estimation algorithm is based on solving a finite horizon "q-period" model in which the continuation value for the q-th period is replaced with its estimate $\hat{V}^0$. Namely, we evaluate the likelihood by applying the fixed point iterations only q-times starting from the initial estimate $\hat{V}^0$ as:

$$\max_{\theta\in\Theta} \quad \sum_{i=1}^{n} \ln \Lambda(\theta, V_\theta^q)(a_i|x_i), \quad \text{where} \quad V_\theta^q = \underbrace{\Gamma(\theta, \Gamma(\theta, ...\Gamma}_{\text{q times}}(\theta, \hat{V}^0))) \equiv \Gamma^q(\theta, \hat{V}^0), \tag{3}$$

where $\Gamma^q(\theta, \cdot)$ is a q-fold operator of $\Gamma(\theta, \cdot)$. Note that $V_\theta^q(x)$ in (3) represents the value of a firm with machine age x when a firm makes an optimal dynamic decision over q-periods where the (q+1)-th period's value is given by $\hat{V}^0$. Solving the optimization problem (3) is much less computationally intensive than implementing the NFXP algorithm.

Given an arbitrary starting value of $\hat{V}^0$, an estimator of θ based on (3) is generally inconsistent. *If an initial estimator $\hat{V}^0$ is consistent*, however, our proposed estimator is consistent. Furthermore, we may apply the idea of the NPL algorithm [cf., Aguirregabiria and Mira (2002, 2007) and Kasahara and Katsumi (2008a,2008b)] in this context to improve the efficiency; namely, once an estimate of θ is obtained from solving the optimization problem (3), one can update the value function estimate $\hat{V}^0$ as $\hat{V}^1 = \Gamma^q(\hat{\theta}, \hat{V}^0)$, which can provide a more accurate estimator of V_θ than $\hat{V}^0$. Next, one can obtain another estimator of θ, $\hat{\theta}^1$, by solving (3) using $\hat{V}^1$ in place of $\hat{V}^0$. Iterating this procedure generates a sequence of estimators $\{\hat{\theta}^j, \hat{V}^j\}_{j=1}^{\infty}$. Upon convergence, the limit of this sequence is independent of the initial estimator $\hat{V}^0$ and, thus, this method is applicable even when an initial consistent estimator for V is not available.

Our proposed estimator is applicable to a wide range of dynamic programming models. For instance, we use our proposed algorithm to estimate a model with borrowing constraint, where the choice set depends on an unobserved state variable. Its implementation is straightforward in terms of computer programming once the value iteration mapping $\Gamma(\theta, V)$ is coded in computer language; unlike the Hotz-Miller type estimators, there is no need to reformulate a fixed point mapping in value function space as that in the space of probability distributions using the policy iteration operator. We may also apply our method to estimate models with unobserved heterogeneity.

3 The Model

This section introduces the class of discrete Markov decision models considered in this paper. An agent maximizes the expected discounted sum of utilities, $E[\sum_{j=0}^{\infty} \beta^j U_\theta(a_{t+j}, s_{t+j})|a_t, s_t]$, where s_t is the vector of states and a_t is a discrete action to be chosen from the constraint set

$G_\theta(s_t) \subset A \equiv \{a^1, a^2, ..., a^{|A|}\}$. The transition probabilities are given by $p_\theta(s_{t+1}|s_t, a_t)$. The Bellman equation for this dynamic optimization problem is written as

$$W(s_t) = \max_{a \in G_\theta(s_t)} \left\{ U_\theta(a, s_t) + \beta \int W(s_{t+1}) p_\theta(s_{t+1}|s_t, a) ds_{t+1} \right\}.$$

From the viewpoint of an econometrician, the state vector can be partitioned as $s_t = (x_t, \epsilon_t, \xi_t)$, where $x_t \in X$ is observable state variable, ϵ_t is serially correlated unobserved state variable, and ξ_t is idiosyncratic unobservable state variable. We make the following assumptions.

Assumption 1 (Conditional Independence of ξ_t): The transition probability function of the state variables can be written as

$$p_\theta(s_{t+1}|s_t, a_t) = g_\theta(\xi_{t+1}|x_{t+1}, \epsilon_{t+1}) f_\theta(x_{t+1}, \epsilon_{t+1}|x_t, \epsilon_t, a_t).$$

Assumption 2 (Finite support for (x, ϵ)): The support of x and ϵ is finite and given by $X = \{x^1, ..., x^{|X|}\}$ and $E = \{\epsilon^1, ..., \epsilon^{|E|}\}$, respectively.

The model's setup covers a finite mixture model with permanent unobserved heterogeneity (when ϵ is permanent unobserved heterogeneity) and a model without serially correlated unobserved state variable (when ϵ is degenerated).

It is assumed that the form of U_θ, G_θ, and f_θ are known up to unknown K-dimensional vector $\theta \in \Theta \subset \mathbb{R}^K$. We are interested in estimating the unknown parameter vector θ from the sample data $\{x_i, a_i\}_{i=1}^n$, where n is the sample size.

Define the integrated value function $V(x, \epsilon) = \int W(x, \epsilon, \xi) g_\theta(\xi|x, \epsilon) d\xi$, and let B_V be the space of $V \equiv \{V(x, \epsilon) : (x, \epsilon) \in X \times E\}$. The Bellman equation can be rewritten in terms of this integrated value function as:

$$V(x, \epsilon) = \int \max_{a \in G_\theta(x, \epsilon, \xi)} \left\{ U_\theta(a, x, \epsilon, \xi) + \beta \sum_{X \times E} V(x', \epsilon') f_\theta(x', \epsilon'|x, \epsilon, a) \right\} g_\theta(\xi'|x, \epsilon) d\xi' \quad (4)$$

Define the Bellman operator defined by the right-hand side of the above Bellman equation as:

$$[\Gamma(\theta, V)](x, \epsilon) \equiv \int \max_{a \in G_\theta(x, \epsilon, \xi)} \left\{ U_\theta(a, x, \epsilon, \xi) + \beta \sum_{X \times E} V(x', \epsilon') f_\theta(x', \epsilon'|x, \epsilon, a) \right\} g_\theta(\xi'|x, \epsilon) d\xi'.$$

The Bellman equation (4) is compactly written as $V = \Gamma(\theta, V)$.

Let $P(a|x, \epsilon)$ denote the conditional choice probabilities of the action a given the state (x, ϵ) and let B_P be the space of $\{P(a|x, \epsilon) : (x, \epsilon) \in X \times E\}$. Given the value function V, $P(a|x, \epsilon)$

is expressed as

$$P(a|x,\epsilon) \quad = \quad \int I\left\{a = \underset{j\in G_\theta(x,\epsilon,\xi)}{\arg\max}\ v_\theta(j,x,\epsilon,\xi,V)\right\} g_\theta(\xi|x,\epsilon)d\xi \tag{5}$$

where

$$v_\theta(a,x,\epsilon,\xi,V) = u_\theta(a,x,\epsilon,\xi) + \beta \sum_{X\times E} V(x',\epsilon')f_\theta(x',\epsilon'|x,\epsilon,a)$$

is the choice-specific value function and $I(\cdot)$ is an indicator function. The right-hand side of the equation (5) can be viewed as a mapping from one Banach (B-) space B_V to another B-space B_P. Define the mapping $\Lambda(\theta,V) : \Theta \times B_V \to B_P$ as

$$[\Lambda(\theta,V)](a|x,\epsilon) \equiv \int I\left\{a = \underset{j\in G_\theta(x,\epsilon,\xi)}{\arg\max}\ v_\theta(j,x,\epsilon,\xi,V)\right\} g_\theta(\xi|x,\epsilon)d\xi. \tag{6}$$

Let θ^0 and P^0 are the true parameter and the true conditional choice probabilities. Define V^0 be the fixed point of $\Gamma(\theta^0,\cdot)$, i.e., $V^0 = \Gamma(\theta^0,V^0)$. Then, the true conditional choice probabilities are related to V^0 as $P^0 = \Lambda(\theta^0,V^0)$.

4 Sequential Estimation

4.1 The model without unobserved heterogeneity

We first consider a class of models without serially correlated unobserved state variables so that

$$[\Gamma(\theta,V)](x) \equiv \int \max_{a\in G_\theta(x,\xi)} \left\{U_\theta(a,x,\xi) + \beta \sum_X V(x')f_\theta(x'|x,a)\right\} g(\xi|x)d\xi,$$

and

$$[\Lambda(\theta,V)](a|x) \equiv \int I\left\{a = \underset{j\in G_\theta(x,\xi)}{\arg\max}\, U_\theta(a,x,\xi) + \beta \sum_X V(x')f_\theta(x'|x,a)\right\} g(\xi|x)d\xi.$$

We assume that we can evaluate $\Gamma(\theta,V)$ and $\Lambda(\theta,V)$. For instance, we can use simulation-based approach.

Consider a cross-sectional data set $\{a_i,x_i\}_{i=1}^n$ where (a_i,x_i) is randomly drawn across i's from the population. The maximum likelihood estimator (MLE) solves the following constrained maximization problem:

$$\max_{\theta\in\Theta} \quad n^{-1}\sum_{i=1}^{n} \ln\{[\Lambda(\theta,V)](a_i|x_i)\}$$
$$s.t. \quad V=\Gamma(\theta,V). \tag{7}$$

The computation of the MLE requires repeatedly solving all the fixed points of $V=\Gamma(\theta,V)$ at each parameter value to maximize the objective function with respect to θ. If evaluating the fixed point of $V=\Gamma(\theta,V)$ is costly, then the MLE is computationally very demanding.

We assume that the support of (a_i,x_i) is finite, $A\times X=\{a^1,a^2,\ldots,a^{|A|}\}\times\{x^1,x^2,\ldots,x^{|X|}\}$. Accordingly, P and V are represented with $L\times 1$ vectors, where $L=|A||X|$. Define a q-fold operator of Γ as

$$\Gamma^q(\theta,V)\equiv \underbrace{\Gamma(\theta,\Gamma(\theta,\ldots\Gamma(\theta,\Gamma}_{q\text{ times}}(\theta,V))\ldots)).$$

Given θ, the Jacobians $\nabla_{V'}\Lambda(\theta,V)$ and $\nabla_{V'}\Gamma^q(\theta,V)$ are $L\times L$ matrices, where $\nabla_{V'}\equiv(\partial/\partial V')$. Define $\Lambda_V\equiv\nabla_{V'}\Lambda(\theta^0,V^0)$, $\Gamma^q_V\equiv\nabla_{V'}\Gamma^q(\theta^0,V^0)$, $\Lambda_\theta\equiv\nabla_{\theta'}\Lambda(\theta^0,V^0)$, and $\Gamma^q_\theta\equiv\nabla_{\theta'}\Gamma^q(\theta^0,V^0)$. Let $\nabla^{(s)}f$ denote the sth order derivative of a function f with respect to its all parameters. Let $\mathcal{N}$ denote a closed neighborhood of (θ^0,V^0), and let $\mathcal{N}_{\theta^0}$ denote a closed neighborhood of θ^0.

Define $\Psi^q(\theta,V)\equiv\Lambda(\theta,\Gamma^q(\theta,V))$. Let $Q_0(\theta,V)\equiv E\ln\Psi^q(\theta,V)(a_i|x_i)$, $\tilde{\theta}_0(V)\equiv\arg\max_{\theta\in\Theta}Q_0(\theta,V)$, and $\phi_0(V)\equiv\Gamma^q(\tilde{\theta}_0(V),V)$. Define the set of population q-NPL fixed points as $\mathcal{Y}_0\equiv\{(\theta,V)\in\Theta\times B_V:\theta=\tilde{\theta}_0(V)\text{ and }V=\phi_0(V)\}$. See AM07 for details.

Assumption 1 *(a) The observations $\{a_i,x_i : i=1,\ldots,n\}$ are independent and identically distributed, and $dF(x)>0$ for any $x\in X$, where $F(x)$ is the distribution function of x_i. (b) $\Psi^q(\theta,V)(a|x)>0$ for any $(a,x)\in A\times X$ and any $(\theta,V)\in\Theta\times B_V$. (c) $\Psi^q(\theta,V)$ is twice continuously differentiable. (d) Θ and B_P are compact. (e) There is a unique $\theta^0\in int(\Theta)$ such that $P^0=\Psi(\theta^0,V^0)$. (f) For any $\theta\neq\theta^0$ and V that solves $V=\Gamma(\theta,V)$, it is the case that $\Psi(\theta,V)\neq P^0$. (g) (θ^0,V^0) is an isolated population q-NPL fixed point. (h) $\tilde{\theta}_0(V)$ is a single-valued and continuous function of V in a neighborhood of V^0. (i) the operator $\phi_0(V)-V$ has a nonsingular Jacobian matrix at V^0.*

Assumption 1(b)(c) implies that $\max_{(a,x)\in A\times X}\sup_{(\theta,V)\in\Theta\times B_V}||\nabla^{(2)}\ln\Psi(\theta,V)(a|x)||<\infty$ and hence $E\sup_{(\theta,V)\in\Theta\times B_V}||\nabla^{(2)}\ln\Psi(\theta,V)(a_i|x_i)||^r<\infty$ for any positive integer r. Assumption 1(h) corresponds to the assumption (iv) in Proposition 2 of AM07. A sufficient condition for Assumption 1(h) is that Q_0 is globally concave in θ in a neighborhood of V^0 and $\nabla_{\theta\theta'}Q_0(\theta,V^0)$ is a nonsingular matrix.

We propose a sequential algorithm that is similar to those proposed by Aguirregabiria and Mira (2002, 2007) and Kasahara and Shimotsu (2008a, 2008b) but, unlike theirs, our algorithm is based on a fixed point mapping defined in the value function space rather than in the probability space. Since it is often difficult to construct a fixed point mapping in the probability space when unobserved state variables are not distributed according to generalized extreme value distribution and the number of choices are larger than three, our proposed method is applicable to a wider class of dynamic programming models than a class of models they consider.

Starting from an initial estimate $\tilde{V}_0$, the q-NPL algorithm iterates the following steps until $j = k$:

Step 1: Given $\tilde{V}_{j-1}$, update θ by

$$\tilde{\theta}_j = \arg\max_{\theta\in\Theta} n^{-1}\sum_{i=1}^{n} \ln\left\{\left[\Lambda(\theta, \Gamma^q(\theta, \tilde{V}_{j-1}))\right](a_i|x_i)\right\}$$

Step 2: Update $\tilde{V}_{j-1}$ using the obtained estimate $\tilde{\theta}_j$: $\tilde{V}_j = \Gamma^q(\tilde{\theta}_j, \tilde{V}_{j-1})$.

Evaluating the objective function for a value of θ involves only q evaluations of the Bellman operator $\Gamma(\theta, \cdot)$ and one evaluation of probability operator $\Lambda(\theta, \cdot)$. The computational cost of Step 1 is roughly equivalent to that of estimating a model with q periods.

This algorithm generates a sequence of estimators $\{\tilde{\theta}_j, \tilde{V}_j\}_{j=1}^{k}$. If this sequence converges, its limit satisfies the following conditions:

$$\check{\theta} = \arg\max_{\theta\in\Theta} n^{-1}\sum_{i=1}^{n} \ln\Lambda(\theta, \Gamma^q(\theta, \check{V}))(a_i|x_i) \quad \text{and} \quad \check{V} = \Gamma^q(\check{\theta}, \check{V}). \tag{8}$$

Any pair $(\check{\theta}, \check{V})$ that satisfies these two conditions in (8) is called an *q-NPL fixed point*. The *q-NPL estimator*, denoted by $(\hat{\theta}_{qNPL}, \hat{V}_{qNPL})$, is defined as the q-NPL fixed point with the highest value of the pseudo likelihood among all the q-NPL fixed points.

Define $\Omega^q_{\theta\theta} \equiv E[\nabla_\theta \ln\Psi^q(\theta^0, V^0)(a_i|x_i)\nabla_{\theta'}\ln\Psi^q(\theta^0, V^0)(a_i|x_i)]$ and $\Omega^q_{\theta V} \equiv E[\nabla_\theta \ln\Psi^q(\theta^0, V^0)(a_i|x_i) \times\nabla_{V'}\ln\Psi^q(\theta^0, V^0)(a_i|x_i)]$. Then, the q-NPL estimator $(\hat{\theta}_{qNPL}, \hat{V}_{qNPL})$ is consistent and its asymptotic distribution is given by: $\sqrt{n}(\hat{\theta}_{NPL} - \theta^0) \to_d N(0, \Sigma_{qNPL})$, where $\Sigma_{qNPL} = [\Omega^q_{\theta\theta} + \Omega^q_{\theta V}(I - \Gamma^q_V)^{-1}\Gamma^q_\theta]^{-1}\Omega^q_{\theta\theta}\{[\Omega^q_{\theta\theta} + \Omega^q_{\theta V}(I - \Gamma^q_V)^{-1}\Gamma^q_\theta]^{-1}\}'$. As q increases, the variance matrix of the q-NPL estimator Σ_{qNPL} approaches that of the MLE.[4]

We now analyze the conditions under which the q-NPL algorithm achieves convergence when started from an initial consistent estimate of V^0, and derive its convergence rate. First, we state the regularity conditions. For matrix and nonnegative scalar sequences of random variables $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$, respectively, we write $X_n = O_p(Y_n)(o_p(Y_n))$ if $||X_n|| \leq CY_n$ for some (all) $C > 0$ with probability arbitrarily close to one for sufficiently large n.

Assumption 2 *Assumption 1 holds. Further, $\tilde{V}_0 - V^0 = o_p(1)$, $\Lambda(\theta, V)$ and $\Gamma(\theta, V)$ are three times continuously differentiable, and $\Omega^q_{\theta\theta}$ is nonsingular.*

[4]The variance of the MLE is given by $(\Omega^\infty_{\theta\theta})^{-1}$. We may show that $\Sigma_{qNPL} \to (\Omega^\infty_{\theta\theta})^{-1}$ as $q \to \infty$ because $\Gamma^q_V \to 0$ as $q \to \infty$.

Define $f_x(x^s) \equiv \Pr(x = x^s)$ for $s = 1, \ldots, |X|$, and let f_x be an $L \times 1$ vector of $\Pr(x = x^s)$ whose elements are arranged conformably with $P_{\theta^0}(a^j|x^s)$. Let $\Delta_P \equiv diag(P^0)^{-1} diag(f_x)$. The following lemma states the local convergence rate of the q-NPL algorithm and is one of the main results of this paper.

Lemma 1 *Suppose Assumption 2 holds. Then, for* $j = 1, \ldots, k$,

$$
\begin{aligned}
\tilde{\theta}_j - \hat{\theta}_{NPL} &= O_p(||\tilde{V}_{j-1} - \hat{V}_{NPL}||), \\
\tilde{V}_j - \hat{V}_{NPL} &= M^q \Gamma_V^q (\tilde{V}_{j-1} - \hat{V}_{NPL}) + O_p(n^{-1/2} ||\tilde{V}_{j-1} - \hat{V}_{NPL}|| + ||\tilde{V}_{j-1} - \hat{V}_{NPL}||^2),
\end{aligned}
$$

where $M^q \equiv I - \Gamma_\theta^q ((\Lambda_V \Gamma_\theta^q + \Lambda_\theta)' \Delta_P (\Lambda_V \Gamma_\theta^q + \Lambda_\theta))^{-1} (\Lambda_V \Gamma_\theta^q + \Lambda_\theta)' \Delta_P \Lambda_V$.

The convergence property of the q-NPL algorithm depends on the dominant eigenvalue of $M^q \Gamma_V^q$. By Blackwell's sufficient conditions, the Bellman operator Γ is a contraction with modulus β, implying that the dominant eigenvalue of Γ_V^q is at most β^q. For sufficiently large q, therefore, the dominant eigenvalue of $M^q \Gamma_V^q$ is less than one in absolute value and the sequence of estimators $\{\tilde{\theta}_j, \tilde{V}_j\}$ converge.

It is possible to reduce the computational burden of implementing the q-NPL algorithm by replacing $\Lambda(\theta, \Gamma^q(\theta, V))$ with its linear approximation around (η, V), where η is a preliminary estimate of θ as follows. Define $\Psi^q(\theta, V) \equiv \Lambda(\theta, \Gamma^q(\theta, V))$ and let $\Psi^q(\theta, V, \eta)$ be a linear approximation of $\Psi^q(\theta, V)$:

$$
[\Psi^q(\theta, V, \eta)](a|x) \equiv [\Psi^q(\eta, V)](a|x) + \{[\nabla_{\theta'} \Psi^q(\eta, V)](a|x)\}(\theta - \eta) \tag{9}
$$

We propose the approximate q-NPL algorithm by replacing $\Psi^q(\theta, V)$ with $\Psi^q(\theta, V, \eta)$ in the first step. Starting from an initial estimate $(\tilde{\theta}_0, \tilde{V}_0)$, the approximate q-NPL algorithm iterates the following steps until $j = k$:

Step 1: Given $(\tilde{\theta}_{j-1}, \tilde{V}_{j-1})$, update θ by

$$
\tilde{\theta}_j = \arg\max_{\theta \in \Theta_j^q} n^{-1} \sum_{i=1}^{n} \ln \left\{ \left[\Psi^q(\theta, \tilde{V}_{j-1}, \tilde{\theta}_{j-1}) \right] (a_i|x_i) \right\},
$$

where $\Theta_j^q \equiv \{\theta \in \Theta : \ \Psi^q(\theta, V, \eta)(a|x) \in [c, 1-c]$ for all $(a, x) \in A \times X\}$ for an arbitrary small $c > 0$.

Step 2: Update $\tilde{V}_{j-1}$ using the obtained estimate $\tilde{\theta}_j$: $\tilde{V}_j = \Gamma^q(\tilde{\theta}_j, \tilde{V}_{j-1})$.

Implementing Step 1 in the approximate q-NPL algorithm is much less computationally intensive than the original q-NPL algorithm because we may evaluate $\Psi^q(\tilde{\theta}_{j-1}, \tilde{V}_{j-1})$ and $\nabla_{\theta'} \Psi^q(\tilde{\theta}_{j-1}, \tilde{V}_{j-1})$ outside of the optimization routine for θ in Step 1. Using one-sided numerical derivatives, evaluating $\nabla_{\theta'} \Psi^q(\tilde{\theta}_{j-1}, \tilde{V}_{j-1})$ requires the $(K+1)q$ function evaluations of $\Gamma(\theta, V)$ and the $(K+1)$

function evaluations of $\Lambda(\theta, V)$. Once $\Psi^q(\tilde{\theta}_{j-1}, \tilde{V}_{j-1})$ and $\nabla_{\theta'}\Psi^q(\tilde{\theta}_{j-1}, \tilde{V}_{j-1})$ are computed, evaluating $\Psi^q(\theta, \tilde{V}_{j-1}, \tilde{\theta}_{j-1})$ across different values of θ's is computationally easy.

The following proposition establishes that the first-order convergence property of the approximate q-NPL algorithm is the same as that of the original q-NPL algorithm.

Proposition 1 *Suppose Assumption holds. Suppose we obtain $\{\tilde{\theta}_j, \tilde{V}_j\}_{j=1}^k$ by the approximate q-NPL algorithm. Then, for $j = 1, \ldots, k$,*

$$\begin{aligned}
\tilde{\theta}_j - \hat{\theta}_{NPL} &= O_p(||\tilde{V}_{j-1} - \hat{V}_{NPL}||) + O_p(n^{-1/2}||\tilde{\theta}_{j-1} - \hat{\theta}|| + ||\tilde{\theta}_{j-1} - \hat{\theta}||^2), \\
\tilde{V}_j - \hat{V}_{NPL} &= M^q\Gamma_V^q(\tilde{V}_{j-1} - \hat{V}_{NPL}) + O_p(n^{-1/2}||\tilde{\theta}_{j-1} - \hat{\theta}|| + ||\tilde{\theta}_{j-1} - \hat{\theta}||^2) \\
&\quad + O_p(n^{-1/2}||\tilde{V}_{j-1} - \hat{V}_{NPL}|| + ||\tilde{V}_{j-1} - \hat{V}_{NPL}||^2),
\end{aligned}$$

where $M^q \equiv I - \Gamma_\theta^q((\Lambda_V\Gamma_\theta^q + \Lambda_\theta)'\Delta_P(\Lambda_V\Gamma_\theta^q + \Lambda_\theta))^{-1}(\Lambda_V\Gamma_\theta^q + \Lambda_\theta)'\Delta_P\Lambda_V$.

Thus, the approximate q-NPL algorithm achieves the same convergence rate in the first order as that of the original q-NPL algorithm.

4.2 The model with permanent unobserved heterogeneity

Suppose that there are M types of agents, where type m is characterized by a type-specific parameter θ^m, and the population probability of being type m is π^m with $\sum_{m=1}^M \pi^m = 1$. These types capture time-invariant state variables that are unobserved by researchers. With a slight abuse of notation, denote $\theta = (\theta^1, \ldots, \theta^M)' \in \Theta^M$ and $\pi = (\pi^1, \ldots, \pi^M)' \in \Theta_\pi$. Then, $\zeta = (\theta', \pi')'$ is the parameter to be estimated, and let $\Theta_\zeta = \Theta^M \times \Theta_\pi$ denote the set of possible values of ζ. The true parameter is denoted by ζ^0.

Consider a panel data set $\{\{a_{it}, x_{it}, x_{i,t+1}\}_{t=1}^T\}_{i=1}^n$ such that $w_i = \{a_{it}, x_{it}, x_{i,t+1}\}_{t=1}^T \in W \equiv (A \times X \times X)^T$ is randomly drawn across i's from the population. The conditional probability distribution of a_{it} given x_{it} for a type m agent is given by $P_{\theta^m} = \Lambda(\theta^m, V_{\theta^m})$, where V_{θ^m} is a fixed point $V_{\theta^m} = \Gamma(\theta^m, V_{\theta^m})$. To simplify our analysis, we assume that the transition probability function of x_{it} is independent of types and given by $f_x(x_{i,t+1}|a_{it}, x_{it})$ and is known to researchers. An extension to the case where the transition probability function is also type-dependent is straightforward.

In this framework, the initial state x_{i1} is correlated with unobserved type (i.e., the initial conditions problem of Heckman (1981)). We assume that x_{i1} for type m is randomly drawn from the type m stationary distribution characterized by a fixed point of the following equation: $p^*(x) = \sum_{x' \in X} p^*(x') \left(\sum_{a' \in A} P_{\theta^m}(a'|x') f_x(x|a', x')\right) \equiv [T(p^*, P_{\theta^m})](x)$. Since solving the fixed point of $T(\cdot, P)$ for given P is often less computationally intensive than computing the fixed point of $\Psi(\cdot, \theta)$, we assume the full solution of the fixed point of $T(\cdot, P)$ is available given P.[5]

[5]It is possible to relax the stationarity assumption on the initial states by estimating the type-specific ini-

Let P^m and V^m denote type m's conditional choice probabilities and type m's value function, stack the P^m's and the V^m's as $\mathbf{P} = (P^{1'}, \dots, P^{M'})'$ and $\mathbf{V} = (V^{1'}, \dots, V^{M'})'$, respectively. Let $\mathbf{P}^0$ and $\mathbf{V}^0$ denote their true values. Let $\mathbf{\Gamma}(\theta, \mathbf{V}) = (\Gamma(\theta^1, V^1)', \dots, \Gamma(\theta^M, V^M)')'$ and let $\mathbf{\Lambda}(\theta, \mathbf{V}) = (\Lambda(\theta^1, V^1)', \dots, \Lambda(\theta^M, V^M)')'$. Then, the maximum likelihood estimator for a model with unobserved heterogeneity is:

$$\begin{aligned} \hat{\zeta}_{MLE} &= \arg\max_{\zeta \in \Theta_\zeta} \ln\left([L(\pi, \mathbf{P})](w_i)\right), \\ \text{s.t.} &\qquad \mathbf{P} = \mathbf{\Lambda}(\theta, \mathbf{V}), \quad \mathbf{V} = \mathbf{\Gamma}(\theta, \mathbf{V}) \end{aligned} \tag{10}$$

where

$$[L(\pi, \mathbf{P})](w_i) = \sum_{m=1}^{M} \pi^m p^*_{P^m}(x_{i1}) \prod_{t=1}^{T} P^m(a_{it}|x_{it}) f_x(x_{i,t+1}|a_{it}, x_{it}),$$

and $p^*_{P^m} = T(p^*_{P^m}, P^m)$ is the type m stationary distribution of x when the conditional choice probability is P^m. If $\mathbf{P}^0 = \Lambda(\theta^0, \mathbf{V}^0)$ is the true conditional choice probability distribution and π^0 is the true mixing distribution, then $L^0 = L(\pi^0, \mathbf{P}^0)$ represents the true probability distribution of w.

We consider the following sequential algorithm for models with unobserved heterogeneity. Let $\mathbf{\Gamma}^q(\theta, V) = (\Gamma^q(\theta^1, V^1)', \dots, \Gamma^q(\theta^M, V^M)')'$. Define $\Psi^q(\theta^m, V^m) = \Lambda(\theta^m, \Gamma^q(\theta^m, V^m))$ for $m = 1, ..., M$ and let $\mathbf{\Psi}^q(\theta, V) = (\Psi^q(\theta^1, V^1)', \dots, \Psi^q(\theta^M, V^M)')'$. Assume that an initial consistent estimator $\tilde{\mathbf{V}}_0 = (\tilde{V}_0^1, \dots, \tilde{V}_0^M)'$ is available. For $j = 1, 2, \dots$, iterate

Step 1: Given $\tilde{\mathbf{V}}_{j-1} = (\tilde{V}_{j-1}^1, \dots, \tilde{V}_{j-1}^M)'$, update $\zeta = (\theta', \pi')'$ by

$$\tilde{\zeta}_j = \arg\max_{\zeta \in \Theta_\zeta} \; n^{-1} \sum_{i=1}^{n} \ln\Big([L(\pi, \mathbf{\Psi}^q(\theta, \tilde{\mathbf{V}}_{j-1}))](w_i)\Big).$$

Step 2: Update $\mathbf{V}$ using the obtained estimate $\tilde{\theta}_j$ by $\tilde{\mathbf{V}}_j = \mathbf{\Gamma}^q(\tilde{\theta}_j, \tilde{\mathbf{V}}_{j-1})$ for $m = 1, ..., M$,

until $j = k$. If iterations converge, its limit satisfies $\hat{\zeta} = \arg\max_{\zeta \in \Theta_\zeta} n^{-1} \sum_{i=1}^{n} \ln([L(\pi, \mathbf{\Psi}^q(\theta, \hat{\mathbf{V}}))](w_i))$ and $\hat{\mathbf{V}} = \mathbf{\Gamma}^q(\hat{\theta}, \hat{\mathbf{V}})$. Among the pairs that satisfy these two conditions, the one that maximizes the pseudo likelihood is called the *NPL estimator*, which we denote by $(\hat{\zeta}_{NPL}, \hat{\mathbf{V}}_{NPL})$.

Let us introduce the assumptions for the consistency and asymptotic normality of the q-NPL estimator. They are analogous to the assumptions used in Aguirregabiria and Mira (2007). define $Q_0(\zeta, \mathbf{V}) \equiv E \ln\left([L(\pi, \mathbf{\Psi}^q(\theta, \mathbf{V}))](w_i)\right)$, $\tilde{\zeta}_0(\mathbf{V}) \equiv \arg\max_{\zeta \in \Theta_\zeta} Q_0(\theta, \mathbf{V})$, and $\phi_0(\mathbf{V}) \equiv \mathbf{\Gamma}^q(\tilde{\theta}_0(\mathbf{V}), \mathbf{V})$. Define the set of population q-NPL fixed points as $\mathcal{Y}_0 \equiv \{(\theta, \mathbf{V}) \in \Theta \times B_V^M : \zeta = \tilde{\zeta}_0(\mathbf{V}) \text{ and } \mathbf{V} = \phi_0(\mathbf{V})\}$.

tial distributions of x, denoted by $\{p^{*m}\}_{m=1}^M$, without imposing stationarity restriction in Step 1 of the q-NPL algorithm. In this case, the q-NPL algorithm has the convergence rate similar to that of Proposition 2.

Assumption 3 *(a) $w_i = \{(a_{it}, x_{it}, x_{i,t+1}) : t = 1, \ldots, T\}$ for $i = 1, \ldots, n$, are independently and identically distributed, and $dF(x) > 0$ for any $x \in X$, where $F(x)$ is the distribution function of x_i. (b) $[L(\pi, \mathbf{P})](w) > 0$ for any w and for any $(\pi, \mathbf{P}) \in \Theta_\pi \times B_P^M$. (c) $\Lambda(\theta, V)$ and $\Gamma(\theta, V)$ are twice continuously differentiable. (d) Θ_ζ and B_P^M are compact. (e) There is a unique $\zeta^0 \in int(\Theta_\zeta)$ such that $[L(\pi^0, \mathbf{P}^0)](w) = [L(\pi^0, \mathbf{\Psi}(\theta^0, \mathbf{V}^0))](w)$. (f) For any $\zeta \neq \zeta^0$ and $\mathbf{V}$ that solves $\mathbf{V} = \mathbf{\Gamma}(\theta, \mathbf{V})$, it is the case that $\Pr(\{w : [L(\pi, \mathbf{\Psi}(\theta, \mathbf{V})](w) \neq L^0(w)\}) > 0$. (g) $(\zeta^0, \mathbf{V}^0)$ is an isolated population q-NPL fixed point. (h) $\tilde{\zeta}_0(\mathbf{V})$ is a single-valued and continuous function of $\mathbf{V}$ in a neighborhood of $\mathbf{V}^0$. (i) the operator $\phi_0(\mathbf{V}) - \mathbf{V}$ has a nonsingular Jacobian matrix at $\mathbf{V}^0$. (j) For any $P \in B_P$, there exists a unique fixed point for $T(\cdot, P)$.*

Under Assumption 3, the consistency and asymptotic normality of the q-NPL estimator can be shown by following the proof of Proposition 2 of Aguirregabiria and Mira (2007).

We now establish the convergence property of the q-NPL algorithm for models with unobserved heterogeneity.

Assumption 4 *Assumption 3 holds. Further, $\tilde{\mathbf{V}}_0 - \mathbf{V}^0 = o_p(1)$, $\Lambda(\theta, V)$ and $\Gamma(\theta, V)$ are three times continuously differentiable, and $\Omega^q_{\zeta\zeta}$ is nonsingular.*

Assumption 4 requires an initial consistent estimator of the value functions. As Aguirregabiria and Mira (2007) argue, if the q-NPL algorithm converges, then the limit may provide a consistent estimate of the parameter ζ even when $\tilde{\mathbf{V}}_0$ is not consistent.

The following proposition states the convergence properties of the q-NPL algorithm for models with unobserved heterogeneity.

Proposition 2 *Suppose Assumptions 3-4 hold. Then, for $j = 1, \ldots, k$,*

$$\begin{aligned}
\tilde{\zeta}_j - \hat{\zeta}_{NPL} &= O_p(||\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}_{NPL}||), \\
\tilde{\mathbf{V}}_j - \hat{\mathbf{V}}_{NPL} &= \mathbf{M}^q \mathbf{\Gamma}^q_V (\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}_{NPL}) \\
&\quad + O_p(n^{-1/2}||\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}_{NPL}||) + O_p(||\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}_{NPL}||^2).
\end{aligned}$$

where

$$\mathbf{M}^q \equiv I - \mathbf{\Gamma}^q_\theta D(\mathbf{\Psi}^q_\theta)' L_P' \Delta_L^{1/2} M_{L_\pi} \Delta_L^{1/2} L_P \mathbf{\Lambda}_V$$

with $D = ((\mathbf{\Psi}^q_\theta)' L_P' \Delta_L^{1/2} M_{L_\pi} \Delta_L^{1/2} L_P \mathbf{\Psi}^q_\theta)^{-1}$, $M_{L_\pi} \equiv I - \Delta_L^{1/2} L_\pi (L_\pi' \Delta_L L_\pi)^{-1} L_\pi \Delta_L^{1/2}$, and $\mathbf{\Psi}^q_\theta \equiv \nabla_{\theta'} \mathbf{\Psi}^q(\theta^0, \mathbf{V}^0)$, $\mathbf{\Gamma}^q_\theta \equiv \nabla_{\theta'} \mathbf{\Gamma}^q(\theta^0, \mathbf{V}^0)$, $\mathbf{\Gamma}^q_V \equiv \nabla_{\mathbf{V}'} \mathbf{\Gamma}^q(\theta^0, \mathbf{V}^0)$, $\mathbf{\Lambda}_V \equiv \nabla_{\mathbf{V}'} \mathbf{\Lambda}(\theta^0, \mathbf{V}^0)$, $\Delta_L = diag((L^0)^{-1})$, $L_P = \nabla_{\mathbf{P}'} L(\pi^0, \mathbf{P}^0)$, and $L_\pi = \nabla_{\pi'} L(\pi^0, \mathbf{P}^0)$.

Since $\mathbf{\Gamma}^q_V \to 0$ as $q \to \infty$, the algorithm is converging for sufficiently large q.

To reduce the computational cost of implementing the q-NPL algorithm, we may apply the approximate q-NPL algorithm to models with unobserved heterogeneity by replacing $\Psi^q(\theta, V)$ with $\Psi^q(\theta, V, \eta)$ in the first step. Let $\eta = (\eta^1, ..., \eta^M)'$ be a preliminary estimate of $\theta =$

$(\theta^1, ..., \theta^M)'$. Let $\mathbf{\Psi}^q(\theta, V, \eta) = (\Psi^q(\theta^1, V^1, \eta^1)', \ldots, \Psi^q(\theta^M, V^M, \eta^1)')'$, where $\Psi^q(\theta, V, \eta)$ is defined in (9). Assume that initial consistent estimators $\tilde{\theta}_0 = (\tilde{\theta}_0^1, \ldots, \tilde{\theta}_0^M)'$ and $\tilde{\mathbf{V}}_0 = (\tilde{V}_0^1, \ldots, \tilde{V}_0^M)'$ are available. The approximate q-NPL algorithm iterates the following steps until $j = k$:

Step 1: Given $\tilde{\theta}_{j-1} = (\tilde{\theta}_{j-1}^1, \ldots, \tilde{\theta}_{j-1}^M)'$ and $\tilde{\mathbf{V}}_{j-1} = (\tilde{V}_{j-1}^1, \ldots, \tilde{V}_{j-1}^M)'$, update $\zeta = (\theta', \pi')'$ by

$$\tilde{\zeta}_j = \underset{\zeta \in \Theta^q_{\zeta,j}}{\arg\max}\ n^{-1} \sum_{i=1}^n \ln \Big([L(\pi, \mathbf{\Psi}^q(\theta, \tilde{\mathbf{V}}_{j-1}, \tilde{\theta}_{j-1}))](w_i) \Big).$$

where $\Theta^q_{\zeta,j} \equiv \{(\pi, \theta) : 0 < \pi^m < 1, [\Psi^q(\theta^m, \tilde{V}_{j-1}^m, \tilde{\theta}_{j-1}^m)](w) \in [c, 1-c]$ for all $w \in W$ for $m = 1, ..., M\}$ for an arbitrary small $c > 0$.

Step 2: Given $(\tilde{\theta}_j, \tilde{\mathbf{V}}_{j-1})$, update $\mathbf{V}$ by $\tilde{\mathbf{V}}_j = \mathbf{\Gamma}^q(\tilde{\theta}_j, \tilde{\mathbf{V}}_{j-1})$ for $m = 1, ..., M$.

If iterations converge, its limit satisfies $\hat{\zeta} = \arg\max_{\zeta \in \Theta_\zeta}\ n^{-1} \sum_{i=1}^n \ln([L(\pi, \mathbf{\Psi}^q(\theta, \hat{\mathbf{V}}))](w_i))$ and $\hat{\mathbf{V}} = \mathbf{\Gamma}^q(\hat{\theta}, \hat{\mathbf{V}})$. Among the pairs that satisfy these two conditions, the one that maximizes the pseudo likelihood is called the *NPL estimator*, which we denote by $(\hat{\zeta}_{NPL}, \hat{\mathbf{V}}_{NPL})$.

The following proposition establishes that the dominant term for the convergence rate of the approximate q-NPL algorithm is the same as that of the q-NPL algorithm for models with unobserved heterogeneity.

Proposition 3 *Suppose Assumptions 3-4 hold. Suppose we obtain $\{\tilde{\zeta}_j, \tilde{V}_j\}_{j=1}^k$ by the approximate q-NPL algorithm. Then, for $j = 1, \ldots, k$,*

$$\begin{aligned}
\tilde{\zeta}_j - \hat{\zeta}_{NPL} &= O_p(||\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}_{NPL}||) + O_p(n^{-1/2}||\tilde{\zeta}_{j-1} - \hat{\zeta}|| + ||\tilde{\zeta}_{j-1} - \hat{\zeta}||^2), \\
\tilde{\mathbf{V}}_j - \hat{\mathbf{V}}_{NPL} &= \mathbf{M}^q \mathbf{\Gamma}_V^q (\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}_{NPL}) + O_p(n^{-1/2}||\tilde{\zeta}_{j-1} - \hat{\zeta}|| + ||\tilde{\zeta}_{j-1} - \hat{\zeta}||^2) \\
&\quad + O_p(n^{-1/2}||\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}_{NPL}|| + ||\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}_{NPL}||^2).
\end{aligned}$$

where $\mathbf{M}^q$ is defined in Proposition 2.

5 Monte Carlo Experiments

5.1 Experimental design

We consider a version of machine replacement model with unobserved heterogeneity as follows. The observed state variable is machine age denoted by $x_t \in \mathbb{N}_x$ and the unobserved state variables include production shock ϵ_t and choice-specific cost shock $\xi_t = (\xi_t(0), \xi_t(1))'$, where ϵ_t is independently drawn from $N(0, \sigma_\epsilon^2)$ while $\xi_t(a)$'s are independently drawn from Type 1 extreme value distribution. The replacement decision is denoted by $a_t \in \{0, 1\}$ and the transition function of x_t is given by $x_{t+1} = a_t + (1 - a_t)(x_t + 1)$. The profit function is given by $u_\theta(x_t, \epsilon_t, a_t) + \xi(a_t)$, where $u_\theta(x_t, \epsilon_t, a_t) = \exp(\theta_1 x_t(1 - a_t) + \epsilon_t) - \theta_2 a_t$.

We assume that $\theta = (\theta_1, \theta_2)$ is multinomially distributed with the number of support points equal to M, where the m-th type is characterized by a type-specific parameter $\theta^m = (\theta_1^m, \theta_2^m)'$ and the fraction of the m-th type in the population is π^m. We also assume that revenue is observable but with measurement error as: $\ln y_t = \theta_1^m x_t(1-a_t)+\epsilon_t+\eta_t$, where η_t is measurement error and is assumed to be independent of ϵ_t and drawn from $N(0, \sigma_\eta^2)$.

The Bellman equation for this firm's dynamic optimization problem is written as

$$V(x,\epsilon) = \gamma + \ln\left(\sum_{a=0}^{1} \exp\left(u_{\theta^m}(x,\epsilon,a) + \beta E_{\epsilon'}[V(x',\epsilon')|x,a]\right)\right) \equiv [\Gamma(\theta^m, V)](x,\epsilon)$$

while the mapping from the value function to the conditional choice probability is given by

$$[\Lambda(\theta^m, V)](a|x,\epsilon) \equiv \frac{\exp\left(u_{\theta^m}(x,\epsilon,a) + \beta E_{\epsilon'}[V(x',\epsilon')|x,a]\right)}{\sum_{a'=0}^{1}\exp\left(u_{\theta^m}(x,\epsilon,a') + \beta E_{\epsilon'}[V(x',\epsilon')|x,a']\right)},$$

where $E_{\epsilon'}[V(x',\epsilon')|x,a] = \int V(a+(1-a)(x+1),\epsilon')\phi(\epsilon'/\sigma_\epsilon)/\sigma_\epsilon d\epsilon'$.

In our experiment, we set $M = 2$ and estimate the five structural parameters $\theta \equiv (\theta^{1'}, \theta^{2'}, \pi^1)'$, of which true value is given by $\theta^1 = (-0.3, 4.0)'$, $\theta^2 = (-0.1, 2.0)'$, and $\pi^1 = \pi^2 = 1/2$. We assume that the other parameters in the model are known and common across unobserved types at $(\beta, \sigma_\epsilon, \sigma_\eta) = (0.96, 0.4, 0.2)$.

We generate a panel data set of sample size n with T periods from a parametric model. We first draw types of firms $\{m_i : i = 1, ..., n\}$ from the multinomial distribution and, then, we draw the initial states $\{(x_{i1}, \epsilon_{i1}) : i = 1, \ldots, n\}$ from the type-specific stationary distributions of (x, ϵ) given θ^{m_i}'s. For firm i, starting from the initial state (x_{i1}, ϵ_{i1}), a_{i1}'s are drawn from the type-specific conditional choice probabilities $P_{\theta^{m_i}}(a|x_{i1}, \epsilon_{i1})$ while η_{i1}'s are simulated to generate y_{i1}'s. Then, starting from the initial state (x_{i1}, a_{i1}), firm i's time-series data is generated from the model under θ^{m_i}. The data set consists of $\{\{(x_{it}, y_{it}, a_{it})\}_{t=1}^T : i = 1, \ldots, n\}$.[6]

To compute the likelihood, let $w_t = \epsilon_t + \eta_t$ and define $\sigma_w^2 = \sigma_\epsilon^2 + \sigma_\eta^2$ and $\rho^2 = \sigma_\epsilon^2/\sigma_w^2$. Then, the density of ϵ conditional on w is given by $g(\epsilon|w) = \phi[(\epsilon - \rho^2 w)/(\sigma_\epsilon\sqrt{1-\rho^2})]/(\sigma_\epsilon\sqrt{1-\rho^2})$, where $\phi(\cdot)$ is the standard normal density function. Denoting the joint density of ϵ and w by $g(\epsilon, w) = g(\epsilon|w)\phi(w/\sigma_w)/\sigma_w$, the firm i's likelihood contribution is computed by integrating out the unobserved heterogeneities, ϵ's and θ^m's, as

$$L(\theta|\{(x_{it}, y_{it}, a_{it})\}_{t=1}^T) = \sum_{m=1}^{m} \pi^m p_{P_{\theta^m}}^*(x_{i1}) \prod_{t=1}^{T} \int P_{\theta^m}(a_{it}|x_{it}, \epsilon') g(\epsilon', \tilde{w}_{it}(\theta^m)) d\epsilon',$$

[6]To simulate the data from the model with a continuous state space, we first solve an approximated model with a discrete state space using a finite number of grids and then use the "self-approximating" property of the Bellman operator [cf., Rust (1996)] to evaluate conditional choice probabilities at points outside of the grids. This allows us to generate a sample with continuously distributed ϵ from the approximated model and to evaluate a likelihood function at points outside of the grids. We approximate the state space of ϵ by 10 grid points using the method of Tauchen (1986) while the state space of x is given by $\{1, \ldots, 20\}$.

where $\tilde{w}_{it}(\theta^m) = \ln y_{it} - \theta_1^m x_{it}(1-a_{it})$ and $p^*_{P_{\theta^m}}(x)$ is the stationary distribution of x implied by the conditional chocie probability P_{θ^m}, where $P_{\theta^m} = \Lambda(\theta, V_{\theta^m})$ given the fixed point $V_{\theta^m} = \Gamma(V_{\theta^m}, \theta^m)$.[7] The maximum likelihood estimator is obtained by maximizing $\sum_{i=1}^n \ln L(\theta|\{(x_{it}, y_{it}, a_{it})\}_{t=1}^T)$.

The q-NPL algorithm is implemented by iterating the following Steps 1 and 2. In Step 1, given $\tilde{V}^m_{j-1}$ for $m = 1, 2$, we update (θ^1, θ^2) by

$$(\tilde{\theta}^1_j, \tilde{\theta}^2_j) = \underset{(\theta^1,\theta^2)\in\Theta^2}{\arg\max}\, n^{-1}\sum_{i=1}^n \ln\left\{\sum_{m=1}^2 \pi^m p^*_{\Psi^q(\theta^m,\tilde{V}^m_{j-1})}(x_{i1})\prod_{t=1}^T \int [\Psi^q(\theta^m, \tilde{V}^m_{j-1})](a_{it}|x_{it},\epsilon')g(\epsilon', \tilde{w}_{it}(\theta^m))d\epsilon'\right\}.$$

Here, $p^*_{\Psi^q(\theta^m,\tilde{V}^m_{j-1})}(x)$ is the stationary distribution of x when a firm follows the decidion rule specified by the choice probabilities $\Psi^q(\theta^m, \tilde{V}^m_{j-1})$. In Step 2, $\tilde{V}^m_{j-1}$'s are updated using $\tilde{\theta}^m_j$'s as $\tilde{V}^m_j = \Gamma^q(\tilde{\theta}^m_j, \tilde{V}^m_{j-1})$ for $m = 1, 2$. The approximate q-NPL algorithm is similarly implemented by replacing $\Psi^q(\theta^m, \tilde{V}^m_{j-1})$ with its linear approximation around $\theta^m = \tilde{\theta}^m_{j-1}$ in Step 1.

5.2 Estimation based on q-NPL and approximate q-NPL algorithm

We first examine the finite sample performance of our proposed estimators based on q-NPL and approximate q-NPL algorithm for $q = 2, 4, 6$, and 8. We simulate 200 samples, each of which consists of $(n, T) = (400, 5)$ observations. To use the q-NPL algorithm, we set the initial value of the expected value function to zeros. Since applying the approximate q-NPL algorithm also requires the initial estimate of θ, we use the q-NPL algorithm at the initial iteration ($k = 1$) to obtain an initial estimate of (θ, V), and then we examine the performance of the approximate q-NPL algorithm starting from the second iteration ($k = 2$).

Table 1 reports the bias and the square roots of the mean squared errors. The bias and the mean squared errors of the estimators from the q-NPL algorithm improve with the number of iterations, k, given the value of $q = 2, 4, 6$, and 8, while they improve with q given the value of k. When k is small, the bias and the mean squared errors of the estimates from the q-NPL algorithm tend to be larger than those of the MLE. The performance of the q-NPL estimators is very similar to that of the q-NPL algorithm across different values of k and q, indicating that our proposed approximation method works in this experiment.

Table 2 reports the average absolute percentage difference between our proposed estimator and the MLE. For both q-NPL estimator and approximate q-NPL estimator, the distance between our proposed estimator and the MLE becomes smaller as k and q increase.

Table 3 shows how the q-NPL estimators after $k = 10$ iterations improve with the sample size across different values of q.

[7] To compute the integral with respect to ϵ given w_i, we approximate the distribution of ϵ conditional on the realized value of w_i for $i = 1, ..., n$ using Tauchen's method.

6 Proofs

6.1 Proof of Lemma 1

Define $\bar{\psi}^q(\theta, V) \equiv n^{-1}\sum_{i=1}^n \ln \Psi^q(\theta, V)(a_i|x_i)$. With these notations, we may write $\Omega^q_{\theta\theta} = (\Psi^q_\theta)'\Delta_P\Psi^q_\theta$ and $\Omega^q_{\theta V} = (\Psi^q_\theta)'\Delta_P\Psi^q_V$, where $\Psi^q_\theta = \Lambda_V\Gamma^q_\theta + \Lambda_\theta$ and $\Psi^q_V = \Lambda_V\Gamma^q_V$.

First, $\tilde{\theta}_j$ satisfies the first order condition $\nabla_\theta\overline{\psi}(\tilde{\theta}_j, \tilde{V}_{j-1}) = 0$. Expanding this around $(\hat{\theta}, \hat{V})$ and using $\nabla_\theta\overline{\psi}(\hat{\theta}, \hat{V}) = 0$ gives

$$0 = \nabla_{\theta\theta'}\overline{\psi}^q(\bar{\theta}, \bar{V})(\tilde{\theta}_j - \hat{\theta}) + \nabla_{\theta V'}\overline{\psi}^q(\bar{\theta}, \bar{V})(\tilde{V}_{j-1} - \hat{V}), \tag{11}$$

where $(\bar{\theta}, \bar{V})$ lie between $(\tilde{\theta}_j, \tilde{V}_{j-1})$ and $(\hat{\theta}, \hat{V})$. It follows from the information matrix equality and the consistency of $(\bar{\theta}, \bar{V})$ that $\nabla_{\theta\theta'}\overline{\psi}(\bar{\theta}, \bar{V}) = -\Omega^q_{\theta\theta} + o_p(1)$ and $\nabla_{\theta V'}\overline{\psi}(\bar{\theta}, \bar{V}) = -\Omega^q_{\theta V} + o_p(1)$. Since $\Omega^q_{\theta\theta}$ is positive definite, we obtain $\tilde{\theta}_j - \hat{\theta} = O_p(||\tilde{V}_{j-1} - \hat{V}||)$, giving the first result.

For the updating equation of V, note that the second derivatives of $\Gamma^q(\theta, V)$ are uniformly bounded in $(\theta, V) \in \Theta \times B_V$ from Assumption. Hence, expanding the right hand side of $\tilde{V}_j = \Gamma^q(\tilde{\theta}_j, \tilde{V}_{j-1})$ twice around $(\hat{\theta}, \hat{V})$ and using $\Gamma^q(\hat{\theta}, \hat{V}) = \hat{V}$, root-$n$ consistency of $(\hat{\theta}, \hat{V})$, and $\tilde{\theta}_j - \hat{\theta} = O_p(||\tilde{V}_{j-1} - \hat{V}||)$, we obtain

$$\tilde{V}_j - \hat{V} = \Gamma^q_\theta(\tilde{\theta}_j - \hat{\theta}) + \Gamma^q_V(\tilde{V}_{j-1} - \hat{V}) + O_p(n^{-1/2}||\tilde{V}_{j-1} - \hat{V}|| + ||\tilde{V}_{j-1} - \hat{V}||^2). \tag{12}$$

Refine (11) as $\tilde{\theta}_j - \hat{\theta} = -\Omega^{-1}_{\theta\theta}\Omega_{\theta V}(\tilde{V}_{j-1} - \hat{V}) + O_p(n^{-1/2}||\tilde{V}_{j-1} - \hat{V}|| + ||\tilde{V}_{j-1} - \hat{V}||^2)$ by using $\nabla_{\theta V'}\overline{\psi}^q(\bar{\theta}, \bar{V}) = -\Omega^q_{\theta V} + O_p(||\tilde{V}_{j-1} - \hat{V}||) + O_p(n^{-1/2})$ and $\nabla_{\theta\theta'}\overline{\psi}^q(\bar{\theta}, \bar{P}) = -\Omega^q_{\theta\theta} + O_p(||\tilde{V}_{j-1} - \hat{V}||) + O_p(n^{-1/2})$. Substituting this into (12) in conjunction with $(\Omega^q_{\theta\theta})^{-1}\Omega^q_{\theta V} = ((\Lambda_V\Gamma^q_\theta + \Lambda_\theta)'\Delta_P(\Lambda_V\Gamma^q_\theta + \Lambda_\theta))^{-1}(\Lambda_V\Gamma^q_\theta + \Lambda_\theta)'\Delta_P\Lambda_V\Gamma^q_V$ gives the stated result. □

6.2 Proof of Proposition 1

We suppress the subscript NPL from $\hat{\theta}_{NPL}$ and $\hat{V}_{NPL}$. Define $\bar{\psi}^q(\theta, V, \eta) \equiv n^{-1}\sum_{i=1}^n \ln \Psi^q(\theta, V, \eta)(a_i|x_i)$. First, expanding the first order condition $0 = \nabla_\theta\psi^q(\tilde{\theta}_j, \tilde{V}_{j-1}, \tilde{\theta}_{j-1})$ twice around $(\hat{\theta}, \tilde{V}_{j-1}, \tilde{\theta}_{j-1})$ gives

$$0 = \nabla_\theta\bar{\psi}^q(\hat{\theta}, \tilde{V}_{j-1}, \tilde{\theta}_{j-1}) + \nabla_{\theta\theta'}\bar{\psi}^q(\hat{\theta}, \tilde{V}_{j-1}, \tilde{\theta}_{j-1})(\tilde{\theta}_j - \hat{\theta}) + O_p(||\tilde{\theta}_j - \hat{\theta}||^2), \tag{13}$$

Second, note that the q-NPL estimator satisfies $\nabla_\theta\bar{\psi}^q(\hat{\theta}, \hat{V}, \hat{\theta}) = 0$, and that $\Psi^q(\theta^0, V^0, \theta^0) = \Psi^q(\theta^0, V^0)$, $\nabla_{\theta'}\Psi^q(\theta^0, V^0, \theta^0) = \nabla_{\theta'}\Psi^q(\theta^0, V^0)$, $\nabla_{V'}\Psi^q(\theta^0, V^0, \theta^0) = \nabla_{V'}\Psi^q(\theta^0, V^0)$, and $\nabla_{\eta'}\Psi^q(\theta^0, P^0, \theta^0) = 0$. Therefore, expanding $\nabla_\theta\bar{\psi}^q(\hat{\theta}, \tilde{V}_{j-1}, \tilde{\theta}_{j-1})$ twice around $(\hat{\theta}, \hat{V}, \hat{\theta})$ and using the root-n consistency of $(\hat{\theta}, \hat{V})$ and the information matrix equality, we obtain $\nabla_\theta\bar{\psi}^q(\hat{\theta}, \tilde{V}_{j-1}, \tilde{\theta}_{j-1}) = -\Omega^q_{\theta V}(\tilde{V}_j - \hat{V}) + r_{nj}$, where r_{nj} denotes a reminder term of $O_p(n^{-1/2}||\tilde{\theta}_{j-1} - \hat{\theta}|| + ||\tilde{\theta}_{j-1} - \hat{\theta}||^2 + n^{-1/2}||\tilde{V}_{j-1} - \hat{V}|| + ||\tilde{V}_{j-1} - \hat{V}||^2)$. Then the stated bound of $\tilde{\theta}_j - \hat{\theta}$ follows from (13) by noting that $\nabla_{\theta\theta'}\psi^q(\hat{\theta}, \tilde{V}_{j-1}, \tilde{\theta}_{j-1}) = -\Omega^q_{\theta\theta} + o_p(1)$.

For the updating equation of V, expanding $\nabla_{\theta\theta'}\bar{\psi}^q(\hat{\theta}, \tilde{P}_{j-1}, \tilde{\theta}_{j-1})$ around $(\hat{\theta}, \hat{P}, \hat{\theta})$ in (13) and using the bound of $\tilde{\theta}_j - \hat{\theta}$ obtained above gives $\tilde{\theta}_j - \hat{\theta} = -(\Omega^q_{\theta\theta})^{-1}\Omega^q_{\theta V}(\tilde{V}_j - \hat{V}) + r_{nj}$. Substituting this into the right hand side of $\tilde{V}_j - \hat{V} = \Gamma^q_\theta(\tilde{\theta}_j - \hat{\theta}) + \Gamma^q_V(\tilde{V}_{j-1} - \hat{V}) + r_{nj}$ and noting $(\Omega^q_{\theta\theta})^{-1}\Omega^q_{\theta V} = ((\Psi^q_\theta)'\Delta_P \Psi^q_\theta)^{-1}(\Psi^q_\theta)'\Delta_P \Psi^q_V = ((\Lambda_V \Gamma^q_\theta + \Lambda_\theta)'\Delta_P(\Lambda_V\Gamma^q_\theta + \Lambda_\theta))^{-1}(\Lambda_V\Gamma^q_\theta + \Lambda_\theta)'\Delta_P \Lambda_V \Gamma^q_V$ gives the stated result. $\square$

6.3 Proof of Proposition 2

We suppress the subscript NPL from $\hat{\zeta}_{NPL}$ and $\hat{\mathbf{V}}_{NPL}$. The proof follows the proof of Lemma 1. Let $l^q(\zeta, \mathbf{V})(w) \equiv \ln(L(\pi, \mathbf{\Psi}^q(\theta, \mathbf{V}))(w))$. Define $\bar{l}^q_\zeta(\zeta, \mathbf{V}) = n^{-1}\sum_{i=1}^n \nabla_\zeta l^q(\zeta, \mathbf{V})(w_i)$, $\bar{l}^q_{\zeta\zeta}(\zeta, \mathbf{V}) = n^{-1}\sum_{i=1}^n \nabla_{\zeta\zeta'} l^q(\zeta, \mathbf{V})(w_i)$, and $\bar{l}^q_{\zeta\mathbf{V}}(\zeta, \mathbf{V}) = n^{-1}\sum_{i=1}^n \nabla_{\zeta\mathbf{V}'} l^q(\zeta, \mathbf{V})(w_i)$. Expanding the first order condition $\bar{l}^q_\zeta(\tilde{\zeta}_j, \tilde{\mathbf{V}}_{j-1}) = \bar{l}^q_\zeta(\hat{\zeta}, \hat{\mathbf{V}}) = 0$ gives

$$\tilde{\zeta}_j - \hat{\zeta} = -\bar{l}^q_{\zeta\zeta}(\bar{\zeta}, \bar{\mathbf{V}})^{-1}\bar{l}^q_{\zeta V}(\bar{\zeta}, \bar{\mathbf{V}})(\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}) = O_p(||\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}||), \tag{14}$$

where $(\bar{\zeta}, \bar{\mathbf{V}})$ is between $(\tilde{\zeta}_j, \tilde{\mathbf{V}}_{j-1})$ and $(\hat{\zeta}, \hat{\mathbf{V}})$. This gives the bound for $\tilde{\zeta}_j - \hat{\zeta}$. Rewriting this further using Assumption 4 gives

$$\tilde{\zeta}_j - \hat{\zeta} = -(\Omega^q_{\zeta\zeta})^{-1}\Omega^q_{\zeta V}(\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}) + O_p(n^{-1/2}||\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}||) + O_p(||\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}||^2), \tag{15}$$

where $\Omega^q_{\zeta\zeta} = E[\nabla_\zeta l^q(\zeta^0, \mathbf{V}^0)(w_i)\nabla_{\zeta'} l^q(\zeta^0, \mathbf{V}^0)(w_i)]$ and $\Omega^q_{\zeta V} = E\left[\nabla_\zeta l^q(\zeta^0, \mathbf{V}^0)(w_i)\nabla_{\mathbf{V}'} l^q(\zeta^0, \mathbf{V}^0)(w_i)\right]$. On the other hand, expanding the second step equation $\tilde{\mathbf{V}}_j = \mathbf{\Gamma}^q(\tilde{\zeta}_j, \tilde{\mathbf{V}}_{j-1})$ twice around $(\hat{\zeta}, \hat{\mathbf{V}})$, using the root-$n$ consistency of $(\hat{\zeta}, \hat{\mathbf{V}})$ and (14) give

$$\tilde{\mathbf{V}}_j - \hat{\mathbf{V}} = \mathbf{\Gamma}^q_V(\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}) + \mathbf{\Gamma}^q_\zeta(\tilde{\zeta}_j - \hat{\zeta}) + O_p(n^{-1/2}||\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}||) + O_p(||\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}||^2), \tag{16}$$

where $\mathbf{\Gamma}^q_\zeta \equiv \nabla_{\zeta'}\mathbf{\Gamma}^q(\theta^0, \mathbf{V}^0) = [\mathbf{\Gamma}^q_\theta, \mathbf{0}]$. Substituting (15) into (16) gives

$$\tilde{\mathbf{V}}_j - \hat{\mathbf{V}} = [\mathbf{\Gamma}^q_V - \mathbf{\Gamma}^q_\zeta(\Omega^q_{\zeta\zeta})^{-1}\Omega^q_{\zeta V}](\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}) + O_p(n^{-1/2}||\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}||) + O_p(||\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}||^2).$$

Note that $\Omega^q_{\zeta\zeta}$ and $\Omega^q_{\zeta V}$ are written as

$$\Omega^q_{\zeta\zeta} = \begin{bmatrix} \Omega^q_{\theta\theta} & \Omega^q_{\theta\pi} \\ \Omega^q_{\pi\theta} & \Omega^q_{\pi\pi} \end{bmatrix} = \begin{bmatrix} (\mathbf{\Psi}^q_\theta)'L_P'\Delta_L L_P \mathbf{\Psi}^q_\theta & (\mathbf{\Psi}^q_\theta)'L_P'\Delta_L L_\pi \\ L_\pi'\Delta_L L_P \mathbf{\Psi}_\theta & L_\pi'\Delta_L L_\pi \end{bmatrix},$$

$$\Omega^q_{\zeta V} = \begin{bmatrix} \Omega^q_{\theta V} \\ \Omega^q_{\pi V} \end{bmatrix} = \begin{bmatrix} (\mathbf{\Psi}^q_\theta)'L_P'\Delta_L L_P \mathbf{\Psi}^q_V \\ L_\pi'\Delta_L L_P \mathbf{\Psi}^q_V \end{bmatrix},$$

and

$$(\Omega^q_{\zeta\zeta})^{-1} = \begin{bmatrix} D & -D\Omega^q_{\theta\pi}(\Omega^q_{\pi\pi})^{-1} \\ -(\Omega^q_{\pi\pi})^{-1}\Omega^q_{\pi\theta}D & (\Omega^q_{\pi\pi})^{-1} + (\Omega^q_{\pi\pi})^{-1}\Omega^q_{\pi\theta}D\Omega^q_{\theta\pi}(\Omega^q_{\pi\pi})^{-1} \end{bmatrix},$$

where $D = ((\boldsymbol{\Psi}^q_\theta)' L_P' \Delta_L^{1/2} M_{L_\pi} \Delta_L^{1/2} L_P \boldsymbol{\Psi}^q_\theta)^{-1}$ with $M_{L_\pi} = I - \Delta_L^{1/2} L_\pi (L_\pi' \Delta_L L_\pi)^{-1} L_\pi \Delta_L^{1/2}$. Then, using $\boldsymbol{\Gamma}^q_\zeta = [\boldsymbol{\Gamma}^q_\theta, \mathbf{0}]$ and $\boldsymbol{\Psi}^q_V = \boldsymbol{\Lambda}_V \boldsymbol{\Gamma}^q_V$ gives $\boldsymbol{\Gamma}^q_\zeta (\Omega^q_{\zeta\zeta})^{-1} \Omega^q_{\zeta P} = \boldsymbol{\Gamma}^q_\theta D (\boldsymbol{\Psi}^q_\theta)' L_P' \Delta_L^{1/2} M_{L_\pi} \Delta_L^{1/2} L_P \boldsymbol{\Lambda}_V \boldsymbol{\Gamma}^q_V$, and the stated result follows. □

6.4 Proof of Proposition 3

We suppress the subscript NPL from $\hat{\zeta}_{NPL}$ and $\hat{\mathbf{V}}_{NPL}$. Let $l^q(\zeta, \mathbf{V}, \eta)(w) \equiv \ln(L(\pi, \boldsymbol{\Psi}^q(\theta, \mathbf{V}, \eta))(w))$. Define $\bar{l}^q_\zeta(\zeta, \mathbf{V}, \eta) = n^{-1} \sum_{i=1}^n \nabla_\zeta l^q(\zeta, \mathbf{V}, \eta)(w_i)$, $\bar{l}^q_{\zeta\zeta}(\zeta, \mathbf{V}, \eta) = n^{-1} \sum_{i=1}^n \nabla_{\zeta\zeta'} l^q(\zeta, \mathbf{V}, \eta)(w_i)$, and $\bar{l}^q_{\zeta\mathbf{V}}(\zeta, \mathbf{V}, \eta) = n^{-1} \sum_{i=1}^n \nabla_{\zeta\mathbf{V}'} l^q(\zeta, \mathbf{V}, \eta)(w_i)$. Note that the q-NPL estimator satisfies $\nabla_\theta \bar{l}^q_\zeta(\hat{\zeta}, \hat{\mathbf{V}}, \hat{\theta}) = 0$ and that $\boldsymbol{\Psi}^q(\zeta^0, \mathbf{V}^0, \theta^0) = \boldsymbol{\Psi}^q(\zeta^0, \mathbf{V}^0)$, $\nabla_{\zeta'} \boldsymbol{\Psi}^q(\zeta^0, \mathbf{V}^0, \theta^0) = \nabla_{\zeta'} \boldsymbol{\Psi}^q(\zeta^0, \mathbf{V}^0)$, $\nabla_{\mathbf{V}'} \boldsymbol{\Psi}^q(\zeta^0, \mathbf{V}^0, \theta^0) = \nabla_{\mathbf{V}'} \boldsymbol{\Psi}^q(\zeta^0, \mathbf{V}^0)$, and $\nabla_{\eta'} \boldsymbol{\Psi}^q(\zeta^0, \mathbf{V}^0, \theta^0) = 0$.

Expanding the first order condition $0 = \bar{l}^q_\zeta(\tilde{\zeta}_j, \tilde{\mathbf{V}}_{j-1}, \tilde{\theta}_{j-1})$ twice around $(\hat{\zeta}, \tilde{\mathbf{V}}_{j-1}, \tilde{\theta}_{j-1})$ gives

$$\begin{aligned} 0 &= \nabla_{\zeta'} \bar{l}^q_\zeta(\hat{\zeta}, \tilde{\mathbf{V}}_{j-1}, \tilde{\theta}_{j-1}) + \nabla_{\zeta\zeta'} \bar{l}^q_\zeta(\hat{\zeta}, \tilde{\mathbf{V}}_{j-1}, \tilde{\theta}_{j-1})(\tilde{\zeta}_j - \hat{\zeta}) + O_p(||\tilde{\zeta}_j - \hat{\zeta}||^2) \\ &= \nabla_{\zeta'} \bar{l}^q_\zeta(\hat{\zeta}, \tilde{\mathbf{V}}_{j-1}, \tilde{\theta}_{j-1}) + \left[-\Omega^q_{\zeta\zeta} + O_p(n^{-1/2} + ||\tilde{\zeta}_{j-1} - \hat{\zeta}|| + ||\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}||)\right](\tilde{\zeta}_j - \hat{\zeta}) + O_p(||\tilde{\zeta}_j - \hat{\zeta}||^2), \end{aligned} \quad (17)$$

where the second equality follows from expanding $\nabla_{\zeta\zeta'} \bar{l}^q_\zeta(\hat{\zeta}, \tilde{\mathbf{V}}_{j-1}, \tilde{\theta}_{j-1})$ around $(\hat{\zeta}, \hat{\mathbf{V}}, \hat{\theta})$ and using the root-n consistency of $(\hat{\zeta}, \hat{\mathbf{V}})$ and the information matrix equality. Furthermore, expanding $\nabla_{\zeta'} \bar{l}^q_\zeta(\hat{\zeta}, \tilde{\mathbf{V}}_{j-1}, \tilde{\theta}_{j-1})$ twice around $(\hat{\zeta}, \hat{\mathbf{V}}, \hat{\theta})$ and using the root-n consistency of $(\hat{\zeta}, \hat{\mathbf{V}})$ and the information matrix equality, we obtain $\nabla_{\zeta'} \bar{l}^q_\zeta(\hat{\zeta}, \tilde{\mathbf{V}}_{j-1}, \tilde{\theta}_{j-1}) = -\Omega^q_{\zeta\mathbf{V}}(\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}) + r_{nj}$, where r_{nj} denotes a reminder term of $O_p(n^{-1/2}||\tilde{\zeta}_{j-1} - \hat{\zeta}|| + ||\tilde{\zeta}_{j-1} - \hat{\theta}||^2 + n^{-1/2}||\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}|| + ||\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}||^2)$. Then, the bound for $\tilde{\zeta}_j - \hat{\zeta}$ follows from writing the second and third terms on the right of (17) as $(-\Omega^q_{\zeta\zeta} + o_p(1))(\tilde{\zeta}_j - \hat{\zeta})$ and using the positive definiteness of $\Omega^q_{\zeta\zeta}$.

For the bound of $\tilde{\mathbf{V}}_j - \hat{\mathbf{V}}$, expanding $\tilde{\mathbf{V}}_j = \boldsymbol{\Gamma}^q(\tilde{\zeta}_j, \tilde{\mathbf{V}}_{j-1})$ twice around $(\hat{\zeta}, \hat{\mathbf{V}})$, using the root-$n$ consistency of $(\hat{\zeta}, \hat{\mathbf{V}})$ and the bound for $\tilde{\zeta}_j - \hat{\zeta}$ give

$$\tilde{\mathbf{V}}_j - \hat{\mathbf{V}} = \boldsymbol{\Gamma}^q_V(\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}) + \boldsymbol{\Gamma}^q_\zeta(\tilde{\zeta}_j - \hat{\zeta}) + O_p(n^{-1/2}||\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}|| + ||\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}||^2). \quad (18)$$

On the other hand, it follows from $\tilde{\zeta}_j - \hat{\zeta} = O_p(||\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}||)$ and (17) that $\tilde{\zeta}_j - \hat{\zeta} = -(\Omega^q_{\zeta\zeta})^{-1}\Omega^q_{\zeta\mathbf{V}}(\tilde{\mathbf{V}}_{j-1} - \hat{\mathbf{V}}) + r_{nj}$. Substituting this into (18) and repeating the argument of Proposition 2 give the stated bound of $\tilde{\mathbf{V}}_j - \hat{\mathbf{V}}$. □

References

Aguirregabiria, V. (1999). "The dynamics of markups and inventories in retailing firms." *Review of Economic Studies* 66: 275-308.

Aguirregabiria, V. and P. Mira (2002). "Swapping the nested fixed point algorithm: a class of estimators for discrete Markov decision models." *Econometrica* 70(4): 1519-1543.

Aguirregabiria, V. and P. Mira (2007). "Sequential estimation of dynamic discrete games." *Econometrica* 75(1): 1-53.

Arcidiacono, P. and R. A. Miller (2008). CCP estimation of dynamic discrete choice models with unobserved heterogeneity. Mimeographed, Duke university.

Bajari, P., Benkard, C. L., and Levin, J. (2007). "Estimating dynamic models of imperfect competition." *Econometrica* 75(5): 1331-1370.

Bajari, P. and H. Hong (2006). Semiparametric estimation of a dynamic game of incomplete information. NBER Technical Working Paper 320.

Collard-Wexler, A. (2006) Demand fluctuations and plant turnover in the Ready-Mix concrete industry. Mimeographed, NYU.

Eckstein, Z. and K. Wolpin (1999). "Why youth drop out of high school: the impact of preferences, opportunities and abilities." *Econometrica* 67(6): 1295-1339.

Gilleskie, D. (1998). "A dynamic stochastic model of medical care use and work absence." *Econometrica* 66: 1-45.

Gotz, G. A. and J. J. McCall (1980). "Estimation in sequential decisionmaking models: a methodological note." *Economics Letters* 6(2): 131-136.

Heckman, J. (1981) "The incidental parameter problem and the problem of initial conditions in estimating a discrete time-discrete data stochastic process," in *Structural Analysis of Discrete Data with Econometric Applications*, ed. by C. Manski and D. McFadden. Cambridge: MIT Press.

Hotz, J. and R. A. Miller (1993). "Conditional choice probabilities and the estimation of dynamic models." *Review of Economic Studies* 60: 497-529.

Kasahara, H. (2009) "Temporary Increases in Tariffs and Investment: The Chilean Case," *Journal of Business and Economic Statistics*, 27(1): 113-127.

Kasahara, H. and B. Lapham (2008). "Productivity and the Decision to Import and Export: Theory and Evidence." CESifo Working Paper No. 2240.

Kasahara, H. and K. Shimotsu (2008a) "Pseudo-likelihood Estimation and Bootstrap Inference for Structural Discrete Markov Decision Models," *Journal of Econometrics*, 146: 92-106.

Kasahara, H. and K. Shimotsu (2008b) Sequential Estimation of Structural Models with a Fixed Point Constraint. Mimeographed, University of Western Ontario.

Miller, R. (1984). "Job matching and occupational choice." *Journal of Political Economy* 92: 1086-1120.

Pakes, A. (1986). "Patents as options: some estimates of the value of holding European patent stocks." *Econometrica* 54: 755-784.

Pakes, A., M. Ostrovsky, and S. Berry (2007). "Simple estimators for the parameters of discrete dynamic games (with entry/exit examples)." *RAND Journal of Economics* 38(2): 373-399.

Pesendorfer, M. and P. Schmidt-Dengler (2008). "Asymptotic least squares estimators for dynamic games," *Review of Economic Studies*, 75, 901-928.

Rothwell, G. and J. Rust (1997). "On the optimal lifetime of nuclear power plants." *Journal of Business and Economic Statistics* 15(2): 195-208.

Rust, J. (1987). "Optimal replacement of GMC bus engines: an empirical model of Harold Zurcher." *Econometrica* 55(5): 999-1033.

Rust, J. (1988). "Maximum likelihood estimation of discrete control processes." *SIAM Journal of Control and Optimization* 26(5): 1006-1024.

Rust, J. (1996). "Numerical dynamic programming in economics." in *Handbook of Computational Economics* eds. H. Amman, D. Kendrick and J. Rust. Elsevier, North Holland.

Wolpin, K. (1984) "An Estimable Dynamic Stochastic Model of Fertility and Child Mortality." *Journal of Political Economy* 92: 852-874.

Table 1: Performance of q-NPL and approximate q-NPL estimator

				q-NPL				approximate q-NPL			
			MLE	q=2	q=4	q=6	q=8	q=2	q=4	q=6	q=8
θ_1^1	Bias	k=1	0.0059	0.0061	0.0060	0.0059	0.0059	0.0061	0.0060	0.0059	0.0059
		k=3	0.0059	0.0056	0.0058	0.0059	0.0059	0.0056	0.0058	0.0059	0.0059
		k=5	0.0059	0.0056	0.0058	0.0059	0.0059	0.0056	0.0058	0.0059	0.0059
		k=10	0.0059	0.0056	0.0058	0.0059	0.0059	0.0056	0.0058	0.0059	0.0059
	$\sqrt{MSE}$	k=1	0.0076	0.0077	0.0077	0.0076	0.0076	0.0077	0.0077	0.0076	0.0076
		k=3	0.0076	0.0074	0.0075	0.0076	0.0076	0.0074	0.0075	0.0076	0.0076
		k=5	0.0076	0.0074	0.0075	0.0076	0.0076	0.0074	0.0075	0.0076	0.0076
		k=10	0.0076	0.0074	0.0075	0.0076	0.0076	0.0074	0.0075	0.0076	0.0076
θ_2^1	Bias	k=1	-0.6887	-1.3293	-0.8081	-0.6748	-0.6774	-1.3293	-0.8081	-0.6748	-0.6774
		k=3	-0.6887	-0.7294	-0.6866	-0.6875	-0.6886	-0.7262	-0.6870	-0.6874	-0.6886
		k=5	-0.6887	-0.7002	-0.6867	-0.6875	-0.6886	-0.6998	-0.6868	-0.6874	-0.6886
		k=10	-0.6887	-0.7014	-0.6868	-0.6875	-0.6886	-0.7014	-0.6868	-0.6874	-0.6886
	$\sqrt{MSE}$	k=1	0.6949	1.3307	0.8118	0.6809	0.6839	1.3307	0.8118	0.6809	0.6839
		k=3	0.6949	0.7343	0.6927	0.6937	0.6947	0.7313	0.6931	0.6936	0.6947
		k=5	0.6949	0.7058	0.6929	0.6937	0.6947	0.7054	0.6930	0.6936	0.6947
		k=10	0.6949	0.7070	0.6929	0.6937	0.6947	0.7070	0.6930	0.6936	0.6947
θ_1^2	Bias	k=1	0.0034	0.0028	0.0033	0.0035	0.0035	0.0028	0.0033	0.0035	0.0035
		k=3	0.0034	0.0032	0.0034	0.0034	0.0034	0.0032	0.0034	0.0034	0.0034
		k=5	0.0034	0.0032	0.0034	0.0034	0.0034	0.0032	0.0034	0.0034	0.0034
		k=10	0.0034	0.0032	0.0034	0.0034	0.0034	0.0032	0.0034	0.0034	0.0034
	$\sqrt{MSE}$	k=1	0.0067	0.0064	0.0066	0.0067	0.0067	0.0064	0.0066	0.0067	0.0067
		k=3	0.0067	0.0066	0.0066	0.0067	0.0067	0.0066	0.0066	0.0067	0.0067
		k=5	0.0067	0.0066	0.0066	0.0067	0.0067	0.0066	0.0066	0.0067	0.0067
		k=10	0.0067	0.0066	0.0066	0.0067	0.0067	0.0066	0.0066	0.0067	0.0067
θ_2^2	Bias	k=1	-0.1568	-0.4253	-0.2114	-0.1552	-0.1530	-0.4253	-0.2114	-0.1552	-0.1530
		k=3	-0.1568	-0.1615	-0.1554	-0.1569	-0.1569	-0.1649	-0.1554	-0.1569	-0.1569
		k=5	-0.1568	-0.1505	-0.1554	-0.1569	-0.1569	-0.1507	-0.1554	-0.1569	-0.1569
		k=10	-0.1568	-0.1515	-0.1554	-0.1569	-0.1569	-0.1515	-0.1554	-0.1569	-0.1569
	$\sqrt{MSE}$	k=1	0.1878	0.4318	0.2319	0.1868	0.1852	0.4318	0.2319	0.1868	0.1852
		k=3	0.1878	0.1903	0.1867	0.1879	0.1879	0.1929	0.1867	0.1879	0.1879
		k=5	0.1878	0.1823	0.1868	0.1879	0.1879	0.1825	0.1867	0.1879	0.1879
		k=10	0.1878	0.1830	0.1868	0.1879	0.1879	0.1830	0.1867	0.1879	0.1879
π^1	Bias	k=1	0.0314	0.0374	0.0324	0.0312	0.0313	0.0374	0.0324	0.0312	0.0313
		k=3	0.0314	0.0315	0.0312	0.0314	0.0314	0.0310	0.0312	0.0314	0.0314
		k=5	0.0314	0.0313	0.0312	0.0314	0.0314	0.0313	0.0312	0.0314	0.0314
		k=10	0.0314	0.0313	0.0312	0.0314	0.0314	0.0313	0.0312	0.0314	0.0314
	$\sqrt{MSE}$	k=1	0.0478	0.0521	0.0484	0.0476	0.0477	0.0521	0.0484	0.0476	0.0477
		k=3	0.0478	0.0478	0.0476	0.0477	0.0478	0.0475	0.0476	0.0477	0.0478
		k=5	0.0478	0.0477	0.0476	0.0477	0.0478	0.0477	0.0476	0.0477	0.0478
		k=10	0.0478	0.0477	0.0476	0.0477	0.0478	0.0477	0.0476	0.0477	0.0478

Notes: Based on 200 simulated samples, each of which consists of $(n, T) = (400, 5)$ observations.

Table 2: Convergence of q-NPL and approximate q-NPL estimator to MLE

		q-NPL				approximate q-NPL			
		q=2	q=4	q=6	q=8	q=2	q=4	q=6	q=8
θ_1^1	k=1	0.0005	0.0002	0.0001	0.0000	0.0005	0.0002	0.0001	0.0000
	k=3	0.0010	0.0003	0.0000	0.0000	0.0010	0.0003	0.0000	0.0000
	k=5	0.0010	0.0003	0.0000	0.0000	0.0010	0.0003	0.0000	0.0000
	k=10	0.0010	0.0003	0.0000	0.0000	0.0010	0.0003	0.0000	0.0000
θ_2^1	k=1	0.1602	0.0299	0.0035	0.0028	0.1602	0.0299	0.0035	0.0028
	k=3	0.0102	0.0005	0.0003	0.0000	0.0094	0.0004	0.0003	0.0000
	k=5	0.0029	0.0005	0.0003	0.0000	0.0028	0.0005	0.0003	0.0000
	k=10	0.0032	0.0005	0.0003	0.0000	0.0032	0.0005	0.0003	0.0000
θ_1^2	k=1	0.0067	0.0009	0.0005	0.0002	0.0067	0.0009	0.0005	0.0002
	k=3	0.0025	0.0006	0.0001	0.0000	0.0025	0.0006	0.0001	0.0000
	k=5	0.0025	0.0006	0.0001	0.0000	0.0025	0.0006	0.0001	0.0000
	k=10	0.0025	0.0006	0.0001	0.0000	0.0025	0.0006	0.0001	0.0000
θ_2^2	k=1	0.1342	0.0273	0.0009	0.0019	0.1342	0.0273	0.0009	0.0019
	k=3	0.0031	0.0007	0.0001	0.0001	0.0042	0.0007	0.0001	0.0001
	k=5	0.0034	0.0007	0.0001	0.0001	0.0033	0.0007	0.0001	0.0001
	k=10	0.0030	0.0007	0.0001	0.0001	0.0030	0.0007	0.0001	0.0001
π^1	k=1	0.0120	0.0019	0.0005	0.0003	0.0120	0.0019	0.0005	0.0003
	k=3	0.0004	0.0005	0.0001	0.0000	0.0009	0.0004	0.0001	0.0000
	k=5	0.0004	0.0005	0.0001	0.0000	0.0004	0.0004	0.0001	0.0000
	k=10	0.0004	0.0005	0.0001	0.0000	0.0004	0.0004	0.0001	0.0000

Notes: The reported values, for instance, are the average of $|(\hat{\theta}^{1,k}_{1,\text{q-NPL}} - \hat{\theta}^1_{1,\text{MLE}})/\theta_1^1|$ across 200 replications.

Table 3: Performance of q-NPL estimator at $k = 10$ for $(n, T) = (200, 5), (400, 5)$, and $(800, 5)$

		q-NPL			
θ_1^1	MLE	q=2	q=4	q=6	q=8
$(n, T) = (200, 5)$	0.0089	0.0088	0.0089	0.0089	0.0089
$(n, T) = (400, 5)$	0.0076	0.0074	0.0075	0.0076	0.0076
$(n, T) = (800, 5)$	0.0067	0.0065	0.0066	0.0067	0.0067
θ_2^1	MLE	q=2	q=4	q=6	q=8
$(n, T) = (200, 5)$	0.6887	0.7005	0.6868	0.6875	0.6886
$(n, T) = (400, 5)$	0.6949	0.7070	0.6929	0.6937	0.6947
$(n, T) = (800, 5)$	0.6853	0.6975	0.6833	0.6841	0.6851
θ_1^2	MLE	q=2	q=4	q=6	q=8
$(n, T) = (200, 5)$	0.0090	0.0090	0.0090	0.0090	0.0090
$(n, T) = (400, 5)$	0.0067	0.0066	0.0066	0.0067	0.0067
$(n, T) = (800, 5)$	0.0052	0.0050	0.0051	0.0052	0.0052
θ_2^2	MLE	q=2	q=4	q=6	q=8
$(n, T) = (200, 5)$	0.2111	0.2070	0.2105	0.2113	0.2111
$(n, T) = (400, 5)$	0.1878	0.1830	0.1868	0.1879	0.1879
$(n, T) = (800, 5)$	0.1827	0.1774	0.1815	0.1828	0.1828
π^1	MLE	q=2	q=4	q=6	q=8
$(n, T) = (200, 5)$	0.0607	0.0605	0.0606	0.0607	0.0607
$(n, T) = (400, 5)$	0.0478	0.0477	0.0476	0.0477	0.0478
$(n, T) = (800, 5)$	0.0389	0.0389	0.0388	0.0389	0.0389